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# A new $Q$-matrix in the eight-vertex model

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**Abstract**
We construct a $Q$-matrix for the eight-vertex model at roots of unity for crossing parameter $\eta = 2mK/L$ with odd $L$, a case for which the existing constructions do not work. The new $Q$-matrix $\hat{Q}$ depends on the spectral parameter $v$ and also on a free parameter $t$. For $t = 0$, $\hat{Q}$ has the standard properties. For $t \neq 0$, however, it does not commute with the operator $S$ nor with itself for different values of the spectral parameter. We show that the six-vertex limit of $\hat{Q}(v, t = \mathrm{i}K'/2)$ exists.

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An essential tool in Baxter's solution of the eight-vertex model [1–4] is the $Q$-matrix which satisfies the $TQ$ equation

$$T(v)Q(v) = [\rho h(v-\eta)]^N Q(v+2\eta) + [\rho h(v+\eta)]^N Q(v-2\eta) \tag{1}$$

and commutes with $T$. Here $T(v)$ is the transfer matrix of the eight-vertex model (A.1). Combined with periodicity properties of $Q(v)$ in the complex $v$-plane equation (1) leads to the derivation of Bethe's equations and the solution of the model. For generic values of the crossing parameter $\eta$ the transfer matrix $T$ has a non-degenerate spectrum. For rational values of $\eta/K$ however this is not the case. This leads to the existence of different $Q$-matrices which all satisfy equation (1). In [1], Baxter constructs a $Q$-matrix valid for

$$2L\eta = 2m_1K + \mathrm{i}m_2K' \tag{2}$$

with integers $m_1, m_2, L$. In [2], Baxter derived a $Q$-matrix valid for generic values of $\eta$. As these $Q$-matrices are different we distinguish them by writing $Q_{72}$ and $Q_{73}$ respectively for the constructions in [1, 2]. It turned out, however, that $Q_{72}$ has interesting properties beyond its role in equation (1) *because* of its restriction to rational values of $\eta/K$.

In [5], it is conjectured that $Q_{72}(v)$ satisfies the following functional relation.

For $N$ even and $\eta = m_1K/L$ where either $L$ even or $L$ and $m_1$ odd

$$\mathrm{e}^{-N\pi \mathrm{i}v/2K} Q_{72}(v - \mathrm{i}K') = A \sum_{l=0}^{L-1} h^N(v - (2l+1)\eta) \frac{Q_{72}(v)}{Q_{72}(v-2l\eta)Q_{72}(v-2(l+1)\eta)}, \tag{3}$$

$A$ is a normalizing constant matrix independent of $v$ that commutes with $Q_{72}$ and $h(v) = H(v)\Theta(v)$. There is a proof of this conjecture valid for $L = 2$ in [6]. This functional relation is important as it allows the conclusion that the dimension of eigenspaces of degenerate eigenvalues of the $T$-matrix is a power of 2, a result also true in the six-vertex model provided the roots of the Drinfeld polynomial of the loop algebra symmetry are distinct [7].

The reason why the case $L$ odd and $m_1$ even is excluded in (3) is that $Q_{72}$ does not exist in this case [5].

The purpose of this paper is to close this gap[1]. We construct for even $N$ a $Q$-matrix which exists for $\eta = 2mK/L$ for odd $L$ which satisfies the functional relation (3). Beyond that we shall show that for $\eta = 2mK/L$ a more general $Q$-matrix exists which depends on a free parameter $t$ and which does not commute with $R$ and $S$ where

$$R = \underbrace{\sigma_1 \otimes \sigma_1 \otimes \cdots \otimes \sigma_1}_{\text{N factors}} \qquad S = \underbrace{\sigma_3 \otimes \sigma_3 \otimes \cdots \otimes \sigma_3}_{\text{N factors}} \tag{4}$$

and not even with itself for different spectral parameters

$$[Q(v_1, t), Q(v_2, t)] \neq 0.$$

This phenomenon has also been observed by Bazhanov and Stroganov in [8] for their column-to-column transfer matrix $T_{\mathrm{col}}$ which acts like a $Q$-matrix in the six-vertex model: it satisfies (1) and it commutes with $T_6$. But it does not commute with itself for different arguments.

We use the notation of Baxter's 1972 paper. We denote our new $Q$-operators by $\hat{Q}_R$, $\hat{Q}_L$ and $\hat{Q}$. They depend on two arguments $v$ and $t$, e.g. $\hat{Q}_R(v, t)$. For brevity we shall write $\hat{Q}_R(v)$ instead of $\hat{Q}_R(v, 0)$. The symbols $Q$, $Q_R$ etc refer to all types of $Q$ matrices.

The plan of this paper is as follows. In section 1 we describe the various steps in the construction of $Q$. We first outline in section 1.1 the general method developed by Baxter and his solution leading to $Q_{72}$. In section 1.2 we present our new $\hat{Q}_R$ operator and describe its range of validity. In section 1.3 we introduce the matrix $Q_L$ and show in section 1.4 that the famous equation $Q_L(u)Q_R(v) = Q_L(v)Q_R(u)$ which Baxter proved for $Q_{72}$ and $Q_{73}$ is also satisfied by $\hat{Q}(v)$. In section 2 we study the quasiperiodicity properties of $Q(v)$ and show that there exists a link between quasiperiodicity of $Q_R$ with quasiperiod i$K'$ and non-existence of $Q_R^{-1}$. We summarize in section 3 the properties of $\hat{Q}(v)$ and describe in section 4 the exotic properties of $\hat{Q}(v, t)$ for $t \neq 0$.

## 1. Construction of a $Q$-matrix for $\eta = 2mK/L$

### *1.1. Baxter's construction of $Q_{72}$*

The goal is to find a matrix $Q_R$ of the form

$$[Q_R(v)]_{\alpha|\beta} = \mathrm{Tr}\, S_R(\alpha_1, \beta_1) S_R(\alpha_2, \beta_2) \cdots S_R(\alpha_N, \beta_N), \tag{5}$$

where $\alpha_j$ and $\beta_j = \pm 1$ and $S_R(\alpha, \beta)$ is a matrix of size $L \times L$ such that $Q_R$ satisfies

$$T(v)Q_R(v) = [\rho h(v - \eta)]^N Q_R(v + 2\eta) + [\rho h(v + \eta)]^N Q_R(v - 2\eta). \tag{6}$$

The $Q$-matrix occurring in equation (1) is then

$$Q(v) = Q_R(v) Q_R^{-1}(v_0) \tag{7}$$

for some constant $v_0$. Therefore, it is necessary that $Q_R(v)$ is a regular matrix. The problem to construct a $Q_R$ of the form (5) satisfying (6) is posed and solved by Baxter in appendix C

[1] There now exists a related investigation by Roan [11].

of [1]. In order to construct a $Q_R$-matrix which is regular for $\eta = mK/L$ for even $m$ and odd $L$, we shall search for other solutions of Baxter's fundamental equations.

These equations are (see (C10), (C11) in [1])

$$\begin{aligned}&(ap_n - bp_m)S_R(+,\beta)_{m,n} + (d - cp_m p_n)S_R(-,\beta)_{m,n} = 0\\&(c - dp_m p_n)S_R(+,\beta)_{m,n} + (bp_n - ap_m)S_R(-,\beta)_{m,n} = 0,\end{aligned} \tag{8}$$

where $\beta = +, -, m, n = 1, \ldots, L$ and $a, b, c, d$ are defined in (A.2). Equations (8) determine the elements of the local matrices $S_R(\alpha, \beta)$ occurring in (5) provided that the determinant of this system of homogeneous linear equations vanishes:

$$(a^2 + b^2 - c^2 - d^2)p_m p_n = ab\big(p_m^2 + pn_n^2\big) - cd\big(1 + p_m^2 p_n^2\big). \tag{9}$$

This determines $p_n$ if $p_m$ is given. Setting

$$p_m = k^{1/2}\,\mathrm{sn}(u), \tag{10}$$

it follows from (A.3) that

$$p_n = k^{1/2}\,\mathrm{sn}(u \pm 2\eta). \tag{11}$$

Baxter selected a solution which has non-vanishing diagonal elements $S_R(\alpha, \beta)_{0,0}$ and $S_R(\alpha, \beta)_{L,L}$. In order to allow $S_R(\alpha, \beta)_{m,n}$ to have non-vanishing diagonal elements $S_R(\alpha, \beta)_{0,0}$ and $S_R(\alpha, \beta)_{L,L}$ equation (9) has to be satisfied for $n = m$. Then

$$\mathrm{sn}(u) = \mathrm{sn}(u \pm 2\eta). \tag{12}$$

This fixes the parameter $u$ to become $u = K \pm \eta$ and leads to the restriction to discrete $\eta$:

$$2L\eta = 2m_1 K + \mathrm{i}m_2 K'. \tag{13}$$

One obtains from (10) and (11) that

$$p_n = k^{1/2}\,\mathrm{sn}(K + (2n-1)\eta) \tag{14}$$

and from (8)

$$\begin{aligned}\mathsf{S}_R(\alpha,\beta)(v)_{k,l} = {}&\delta_{k+1,l}u^\alpha(v + K - 2k\eta)\tau_{-k,\beta} + \delta_{k,l+1}u^\alpha(v + K + 2l\eta)\tau_{l,\beta}\\&+ \delta_{k,1}\delta_{l,1}u^\alpha(v + K)\tau_{0,\beta} + \delta_{k,L}\delta_{l,L}u^\alpha(v + K + 2L\eta)\tau_{L,\beta},\end{aligned} \tag{15}$$

for $1 < k \leqslant L$, $1 < l \leqslant L$ and where

$$u^+(v) = \mathsf{H}(v) \qquad u^-(v) = \Theta(v) \tag{16}$$

if

$$\eta = m_1 K/L. \tag{17}$$

$Q_{R,72}$ is the matrix $Q_R$ defined in (5) with $S_R$ given by (15).

It has been shown in [5] that $Q_R$ based on (15) is singular if $m_1$ is even and $L$ is odd. In the following subsection we show that an alternative construction leads for these $\eta$-values to a regular $Q_R$-matrix.

### *1.2. Another Q-matrix*

To obtain another solution $\hat{S}_R$ of (8) and (9) we consider the possibility that the elements of $\hat{S}_R(\alpha, \beta)_{m,n}$ form cycles

$$\hat{S}_R(\alpha,\beta)_{1,2}, \hat{S}_R(\alpha,\beta)_{2,3}, \ldots, \hat{S}_R(\alpha,\beta)_{L-1,L}, \hat{S}_R(\alpha,\beta)_{L,1}$$

and

$$\hat{S}_R(\alpha,\beta)_{2,1}, \hat{S}_R(\alpha,\beta)_{3,2}, \ldots, \hat{S}_R(\alpha,\beta)_{L,L-1}, \hat{S}_R(\alpha,\beta)_{1,L}$$

instead of imposing the condition that $\hat{S}_R(\alpha, \beta)_{m,n}$ has two diagonal elements. In this case a set of functions $p_n$ consistent with (10) and (11) is

$$p_n = k^{1/2}\,\mathrm{sn}(t + (2n-1)\eta). \tag{18}$$

From the condition that

$$\hat{S}_R(\alpha, \beta)_{L,L+1} = \hat{S}_R(\alpha, \beta)_{L,1} \tag{19}$$

it follows that $p_1$ and $p_L$ must have arguments which differ by $2\eta$:

$$\mathrm{sn}(t + (2L-1)\eta) = \mathrm{sn}(t + \eta - 2\eta). \tag{20}$$

This is satisfied if

$$2L\eta = 4mK + 2\mathrm{i}m_2K'. \tag{21}$$

This condition differs from (13). The solution of equations (8) with the set of $p_n$-functions (18) as input is

$$\hat{S}_R(\alpha, \beta)_{k,l} = \delta_{k+1,l}w^{\alpha}(v - t - 2k\eta)\tau_{\beta,-k} + \delta_{k,l+1}u^{\alpha}(v + t + 2l\eta)\tau_{\beta,l} \tag{22}$$

with $u^{\alpha}$ defined in equation (16) and $w^{\alpha}$ is given by

$$w^{+}(v) = -\mathrm{H}(v) \qquad w^{-}(v) = \Theta(v). \tag{23}$$

Note that the first component of $w^{\alpha}$ differs from $u^{+}$ by a minus sign.

We consider only the case $m_2 = 0$ in (21). Then

$$\eta = 2mK/L. \tag{24}$$

We shall denote the $Q_R$-, $Q_L$- and $Q$-matrices derived from $\hat{S}_R$, $\hat{S}_L$ by $\hat{Q}_R$, $\hat{Q}_L$ and $\hat{Q}$. We distinguish the following cases.

(1) If $L$ is odd the resulting $\hat{Q}$-matrices cover exactly the set of discrete $\eta$-values which is missing in the original solution (15)–(16). We note that for $t = K$ this solution becomes identical to case (15)–(16) with singular $\hat{Q}_R$. But for generic $t$ (especially $t = 0$) $\hat{Q}_R$ is regular. It must be stressed, however, that the regularity has not been proved analytically but numerically for sufficiently large systems to allow the occurrence of degenerate eigenvalues of the transfer matrix $T$. See also appendix C of [1, 5].

(2) $L$ is even but both $L_1 = L/2$ and $m$ are odd.
Then $\eta = mK/L_1$ is that set of $\eta$-values for which solution (15)–(16) leads to regular $Q_R$ matrices. It turns out that in this case the $\hat{Q}_R$-matrix resulting from solution (22) is singular.

(3) $L$ and $L/2$ are even and $m$ is odd.
In this case both solutions (15)–(16) and (22) give regular $Q_R$-matrices. But the matrices $\hat{S}_R(\alpha, \beta)$ differ in size by a factor of 2.

The conclusion is that the two sets of $Q$-matrices (15)–(16) and (22) are complementary in the sense that for $\eta = mK/L$ and odd $L$ what is missing in the first set is present in the second and vice versa.

### *1.3. The matrix $\hat{Q}_L$*

To get finally a $Q$-matrix which commutes with the transfer matrix $T$ and satisfies equation (1) Baxter introduced a second matrix $Q_L$. By transposing equation (6) and replacing $v$ by $-v$ one obtains

$$Q_L(v)T(v) = [\rho h(v-\eta)]^N Q_L(v+2\eta) + [\rho h(v+\eta)]^N Q_L(v-2\eta) \tag{25}$$

with

$$Q_L(v) = Q_R^t(-v) \tag{26}$$

and

$$[Q_L(v)]_{\alpha|\beta} = \mathrm{Tr}\, S_L(\alpha_1, \beta_1) S_L(\alpha_2, \beta_2) \cdots S_L(\alpha_N, \beta_N). \tag{27}$$

We perform this construction for the new $\hat{Q}_R$ matrix. The local matrices $\hat{S}_L$ are obtained from (22)

$$\hat{S}_L(\alpha, \beta)_{k,l}(v) = \hat{S}_R(\beta, \alpha)_{k,l}(-v) \tag{28}$$

$$\hat{S}_L(\alpha, \beta)_{k,l} = \delta_{k+1,l}\tau_{\alpha,-k} u^\beta(v + t + 2k\eta) + \delta_{k,l+1}\tau_{\alpha,l} w^\beta(v - t - 2l\eta). \tag{29}$$

### *1.4. The relation* $Q_L(u)Q_R(v) = Q_L(v)Q_R(u)$

To prove that the $Q$-matrix defined by

$$Q(v) = Q_R(v) Q_R^{-1}(v_0) \tag{30}$$

commutes with the transfer matrix $T$ Baxter shows in [1] that the relation

$$Q_L(v) Q_R(u) = Q_L(u) Q_R(v) \tag{31}$$

holds. Then

$$Q(v) = Q_L^{-1}(u) Q_L(v) = Q_R(v) Q_R^{-1}(u) \tag{32}$$

commutes with $T(v)$. To prove (31) it is shown in [1] that $S_L(\alpha, \gamma)_{m,n}(u) S_R(\gamma, \beta)_{m'n'}(v)$ and $S_L(\alpha, \gamma)_{m,n}(v) S_R(\gamma, \beta)_{m'n'}(u)$ are related by a similarity transformation:

$$S_L(\alpha, \gamma)_{m,n}(u) S_R(\gamma, \beta)_{m'n'}(v) = Y_{m,m';k,k'} S_L(\alpha, \gamma)_{k,l}(v) S_R(\gamma, \beta)_{k'l'}(u) Y^{-1}_{l,l';n,n'}, \tag{33}$$

with diagonal matrix $Y$,

$$Y_{m,m';k,k'} = y_{m,m'} \delta_{m,k} \delta_{m',k'}. \tag{34}$$

To investigate whether the matrices $\hat{Q}_R$ and $\hat{Q}_L$ defined in (5), (22) and (27), (29) fulfil such a relation we define a series of abbreviations. According to (22) we write

$$\hat{S}_R(\alpha, \beta)_{m,n} = \Phi^\alpha_{m,n} \bar{\tau}^\beta_{m,n}, \tag{35}$$

where

$$\Phi^\alpha_{m,n} = \epsilon^\alpha_{m,n} f^\alpha(v_{m,n}) \tag{36}$$

$$v_{m,n} = \delta_{m-1,n}(v + t + 2n\eta) + \delta_{m+1,n}(v - t - 2m\eta) \tag{37}$$

$$\epsilon^\alpha_{m,n} = \delta_{m-1,n} - \alpha\delta_{m+1,n} \qquad \alpha = \pm 1 \tag{38}$$

$$\bar{\tau}^\beta_{m,n} = \delta_{m-1,n}\tau_{\beta,n} + \delta_{m+1,n}\tau_{\beta,-m} \tag{39}$$

and $f^+(v) = H(v)$, $f^-(v) = \Theta(v)$, $\delta_{m+L,n} = \delta_{m,n}$.

Equivalently, we write following (29):

$$\hat{S}_L(\alpha, \beta)_{m,n} = \tau'^\alpha_{m,n} \chi^\beta_{m,n}, \tag{40}$$

where

$$\chi^\beta_{m,n} = \lambda^\beta f^\beta(u_{m,n}) \tag{41}$$

$$u_{m,n} = \delta_{m-1,n}(v - t - 2n\eta) + \delta_{m+1,n}(v + t + 2m\eta) \tag{42}$$

$$\lambda^{\beta}_{m,n} = -\beta\delta_{m-1,n} + \delta_{m+1,n} \tag{43}$$

$$\bar{\tau'}^{\alpha}_{m,n} = \delta_{m-1,n}\tau'_{\alpha,n} + \delta_{m+1,n}\tau'_{\alpha,-m}. \tag{44}$$

It follows then from (35) and (40)

$$\hat{S}_L(\alpha,\gamma)_{m,n}(u)\hat{S}_R(\gamma,\beta)_{m'n'}(v) = \tau'^{\alpha}_{m,n}\chi^{\gamma}_{m,n}(u)\Phi^{\gamma}_{m',n'}(v)\tau^{\beta}_{m',n'} \tag{45}$$

and from (36) and (41) one obtains

$$\begin{aligned}\chi^{\gamma}_{m,n}(u)\Phi^{\gamma}_{m',n'}(v) = {} & (\delta_{m+1,n}\delta_{m'+1,n'} + \delta_{m-1,n}\delta_{m'-1,n'})(\Theta(u_{m,n})\Theta(v_{m',n'}) - H(u_{m,n})H(v_{m',n'})) \\ & + (\delta_{m+1,n}\delta_{m'-1,n'} + \delta_{m-1,n}\delta_{m'+1,n'})(\Theta(u_{m,n})\Theta(v_{m',n'}) + H(u_{m,n})H(v_{m',n'})),\end{aligned} \tag{46}$$

with non-vanishing elements

$$\chi^{\gamma}_{m,m+1}(u)\Phi^{\gamma}_{m',m'+1}(v) = \Theta(u_{m,m+1})\Theta(v_{m',m'+1}) - H(u_{m,m+1})H(v_{m',m'+1}) \tag{47}$$

$$\chi^{\gamma}_{m,m-1}(u)\Phi^{\gamma}_{m',m'-1}(v) = \Theta(u_{m,m-1})\Theta(v_{m',m'-1}) - H(u_{m,m-1})H(v_{m',m'-1}) \tag{48}$$

$$\chi^{\gamma}_{m,m+1}(u)\Phi^{\gamma}_{m',m'-1}(v) = \Theta(u_{m,m+1})\Theta(v_{m',m'-1}) + H(u_{m,m+1})H(v_{m',m'-1}) \tag{49}$$

$$\chi^{\gamma}_{m,m-1}(u)\Phi^{\gamma}_{m',m'+1}(v) = \Theta(u_{m,m-1})\Theta(v_{m',m'+1}) + H(u_{m,m-1})H(v_{m',m'+1}). \tag{50}$$

The arguments are

$$\begin{aligned} u_{m,m+1}(u) - v_{m',m'+1}(v) &= u - v + 2(m+m')\eta + 2t \\ u_{m,m-1}(u) - v_{m',m'-1}(v) &= u - v - 2(n+n')\eta - 2t \\ u_{m,m+1}(u) - v_{m',m'-1}(v) &= u - v + 2(m-m'+1)\eta \\ u_{m,m-1}(u) - v_{m',m'+1}(v) &= u - v - 2(m-m'-1)\eta \\ u_{m,m+1}(u) + v_{m',m'+1}(v) &= u + v + 2(m-m')\eta \\ u_{m,m-1}(u) + v_{m',m'-1}(v) &= u + v + 2(-n+n')\eta \\ u_{m,m+1}(u) + v_{m',m'-1}(v) &= u + v + 2(m+n')\eta + 2t \\ u_{m,m-1}(u) + v_{m',m'+1}(v) &= u + v - 2(n+m')\eta - 2t. \end{aligned} \tag{51}$$

To rewrite (47)–(50), we use

$$\Theta(u)\Theta(v) + H(u)H(v) = cf_+(u+v)g_+(u-v) \tag{52}$$

$$\Theta(u)\Theta(v) - H(u)H(v) = cf_-(u+v)g_-(u-v) \tag{53}$$

$$f_+(u) = H((\mathrm{i}K'+u)/2)H((\mathrm{i}K'-u)/2)g_+(u) = H_1((\mathrm{i}K'+u)/2)H_1((\mathrm{i}K'-u)/2) \tag{54}$$

$$f_-(u) = H_1((\mathrm{i}K'+u)/2)H_1((\mathrm{i}K'-u)/2)g_-(u) = H((\mathrm{i}K'+u)/2)H((\mathrm{i}K'-u)/2). \tag{55}$$

We need especially the following properties of $g_\pm$:

$$g_\pm(-u) = g_\pm(u) \qquad g_\pm(u+4K) = g_\pm(u). \tag{56}$$

After insertion of (52)–(55) into (47)–(50) we get

$$\chi^{\gamma}_{m,m+1}(u)\Phi^{\gamma}_{m',m'+1}(v) = cf_-(u+v+2(m-m')\eta)g_-(u-v+2(m+m')\eta+2t) \tag{57}$$

$$\chi^{\gamma}_{m,m-1}(u)\Phi^{\gamma}_{m',m'-1}(v) = cf_-(u+v+2(m'-m)\eta)g_-(u-v-2(n+n')\eta-2t) \tag{58}$$

$$\chi^{\gamma}_{m,m+1}(u)\Phi^{\gamma}_{m',m'-1}(v) = cf_+(u+v+2(m+n')\eta+2t)g_+(u-v+2(m-m'+1)\eta) \tag{59}$$

$$\chi^{\gamma}_{m,m-1}(u)\Phi^{\gamma}_{m',m'+1}(v) = cf_+(u+v-2(n+m')\eta-2t)\eta)g_+(u-v-2(m-m'-1)\eta). \tag{60}$$

It now remains to show that a $L^2 \times L^2$ matrix $Y$ exists such that equation (33) is satisfied for $\hat{S}_R$ and $\hat{S}_L$. As $\tau$ and $\tau'$ occurring in the definition of $\hat{S}_R$ and $\hat{S}_L$ are free parameters we obtain from (33)

$$\chi^{\gamma}_{m,n}(u)\Phi^{\gamma}_{m',n'}(v) = Y_{m,m';k,k'}\chi^{\gamma}(v)_{k,l}\Phi^{\gamma}(u)_{k',l'}Y^{-1}_{l,l';n,n'}. \tag{61}$$

Taking tentatively $Y$ to be diagonal

$$Y_{m,m';k,k'} = y_{m,m'}\delta_{m,k}\delta_{m',k'}, \tag{62}$$

we get

$$\chi^{\gamma}_{m,n}(u)\Phi^{\gamma}_{m',n'}(v) = \frac{y_{m,m'}}{y_{n,n'}}\chi^{\gamma}_{m,n}(v)\Phi^{\gamma}_{m',n'}(u) \tag{63}$$

and it follows from (57)–(60)

$$y_{m+1,m'+1} = y_{m,m'}\frac{g_-(u-v-2(m+m')\eta-2t)}{g_-(u-v+2(m+m')\eta+2t)} \tag{64}$$

$$y_{m+1,m'-1} = y_{m,m'}\frac{g_+(u-v-2(m-m'+1)\eta)}{g_+(u-v+2(m-m'+1)\eta))}. \tag{65}$$

To prove that a matrix $Y$ can be found such that (61) is satisfied we have to show that the set of equations (64)–(65) is free from contradictions on the torus of size $L \times L$ where

$$y_{m+L,n+L} = y_{m,n}. \tag{66}$$

It follows from equation (64) that

$$\begin{aligned} y_{m+L,n+L} = {} & \frac{g_-(u-v-2(m+n)\eta-4(L-1)\eta-2t)}{g_-(u-v+2(m+n)\eta+4(L-1)\eta+2t)} \\ & \times \frac{g_-(u-v-2(m+n)\eta-4(L-2)\eta-2t)}{g_-(u-v+2(m+n)\eta+4(L-2)\eta+2t)} \cdots \\ & \frac{g_-(u-v-2(m+n)\eta-2t)}{g_-(u-v+2(m+n)\eta+2t)} y_{m,n}. \end{aligned} \tag{67}$$

The factor $g_-(u-v-2(m+n)\eta+4r_2\eta-2t)$ in the numerator cancels the factor $g_-(u-v+2(m+n)\eta+4r_1\eta+2t)$ in the denominator if $t=0$ and

$$-2(m+n)\eta - 4r_2\eta = 2(m+n)\eta + 4r_1\eta + 4kK \tag{68}$$

for arbitrary $k$ and if we set $k = 2m_1k_1$ for integer $k_1$:

$$r_2 = k_1L - m - n - r_1. \tag{69}$$

It follows that for each factor in the numerator of equation (67) there is a factor in the denominator against which it cancels. Similarly we derive from equation (65) that

$$y_{m,n} = y_{m-L,n+L}. \tag{70}$$

We have shown that all $y_{m,n}$ can be determined from a single element (e.g. $y_{1,1}$) consistently if $t=0$. This conclusion cannot be drawn for $t \neq 0$. A numerical test of (31) shows that it is not satisfied for $t \neq 0$, and therefore no similarity transformation (33) exists for $t \neq 0$.

We summarize what has been found in this section.

We have attained our goal to construct a $Q$-matrix which exists for $\eta = 2mK/L$ for odd $L$:

The $\hat{Q}_R$-matrix defined in equation (5) with local matrices $\hat{S}_R$ defined in (22) is regular.

If the parameter $t$ is set to zero relation (31) is satisfied.

Then $\hat{Q}(v) = \hat{Q}_R(v)\hat{Q}_R^{-1}(v_0)$ satisfies equation (1) and commutes with the transfer matrix $T$.

## 2. Quasiperiodicity properties of $Q$

It is easily seen that $Q_{72,R}(v)$ and $\hat{Q}_R(v,t)$ satisfy

$$\hat{Q}_{72,R}(v+2K) = S\hat{Q}_{72,R}(v) \qquad \hat{Q}_R(v+2K,t) = S\hat{Q}_R(v,t). \tag{71}$$

It is of great importance to find the quasiperiodicity properties of the $Q$-matrices in the complex $v$-plane. We do that in this section for $Q_{72,R}(v)$ and $\hat{Q}_R(v, t=0)$. It is well known that the quasiperiod of $Q_{73}$ is $\mathrm{i}\mathsf{K}'$. See [9] for details. We shall present plausibility arguments for the statement that $Q_{72,R}$ as well as $\hat{Q}$ are singular matrices if their quasiperiod is $\mathrm{i}\mathsf{K}'$.

### *2.1. Quasiperiodicity properties of $Q_{72}$*

We get from equations (15), (A.4), (A.5) and from

$$\eta = m_1 K/L \tag{72}$$

the relations

$$\begin{aligned}
\mathsf{S}_R(\pm,\beta)_{k,k+1}(v+\mathrm{i}\mathsf{K}') &= f(v)\exp(+\mathrm{i}\pi k\eta/K)\mathsf{S}_R(\mp,\beta)(v)_{k,k+1}\\
\mathsf{S}_R(\pm,\beta)_{k+1,k}(v+\mathrm{i}\mathsf{K}') &= f(v)\exp(-\mathrm{i}\pi k\eta/K)\mathsf{S}_R(\mp,\beta)(v)_{k+1,k}\\
\mathsf{S}_R(\pm,\beta)_{1,1}(v+\mathrm{i}\mathsf{K}') &= f(v)\mathsf{S}_R(\mp,\beta)(v)_{1,1}\\
\mathsf{S}_R(\pm,\beta)_{L,L}(v+\mathrm{i}\mathsf{K}') &= (-1)^{m_1}f(v)\mathsf{S}_R(\mp,\beta)(v)_{L,L},
\end{aligned} \tag{73}$$

where

$$f(v) = q^{-1/4}\exp\left(-\frac{\mathrm{i}\pi v}{2K}\right). \tag{74}$$

The similarity transformation

$$\bar{S}(\alpha,\beta)_{i,l} = A_{i,j}\hat{S}(\alpha,\beta)_{j,k}A^{-1}_{k,l}, \tag{75}$$

with

$$A_{k,l} = \delta_{k,l}\exp\left(\frac{\mathrm{i}\pi}{2K}(k-1)k\eta\right)a_1 \tag{76}$$

leads to

$$\begin{aligned}
\tilde{\mathsf{S}}_R(\pm,\beta)_{k,k+1}(v+\mathrm{i}\mathsf{K}') &= f(v)\mathsf{S}_R(\mp,\beta)(v)_{k,k+1}\\
\tilde{\mathsf{S}}_R(\pm,\beta)_{k+1,k}(v+\mathrm{i}\mathsf{K}') &= f(v)\mathsf{S}_R(\mp,\beta)(v)_{k+1,k}\\
\tilde{\mathsf{S}}_R(\pm,\beta)_{1,1}(v+\mathrm{i}\mathsf{K}') &= f(v)\mathsf{S}_R(\mp,\beta)(v)_{1,1}\\
\tilde{\mathsf{S}}_R(\pm,\beta)_{L,L}(v+\mathrm{i}\mathsf{K}') &= (-1)^{m_1}f(v)\mathsf{S}_R(\mp,\beta)(v)_{L,L}.
\end{aligned} \tag{77}$$

If $m_1$ is even it follows that

$$\tilde{\mathsf{S}}_R(\alpha,\beta)_{k,l}(v+\mathrm{i}\mathsf{K}') = f(v)R(\alpha,\gamma)\mathsf{S}_R(\gamma,\beta)(v)_{k,l}, \tag{78}$$

where $R$ is defined in equation (4) and

$$Q_{R,72}(v + \mathrm{i}K') = f(v)^N R Q_{R,72}(v). \tag{79}$$

However, it is well known [5] that for even $m_1$ $Q_{R,72}(v)$ is singular and consequently relation (79) cannot be upgraded from $Q_{R,72}$ to $Q_{72}$. It is shown in [5] that instead of (79) the following relation holds:

$$Q_{R,72}(v + 2\mathrm{i}K') = q^{-N} \exp(-\mathrm{i}N\pi v/K) Q_{R,72}(v), \tag{80}$$

which is correct for all $\eta = m_1 K/L$. Provided $m_1$ is odd it follows

$$Q_{72}(v + 2\mathrm{i}K') = q^{-N} \exp(-\mathrm{i}N\pi v/K) Q_{72}(v). \tag{81}$$

### 2.2. *Quasiperiodicity properties of* $\hat{Q}$

We obtain from equation (22)

$$\hat{S}_R(\pm, \beta)(v + \mathrm{i}K')_{k,k+1} = f(v)(-\mathrm{i}) \exp(+\mathrm{i}\pi k\eta/K) \hat{S}_R(\mp, \beta)(v)_{k,k+1} \tag{82}$$

$$\hat{S}_R(\pm, \beta)(v + \mathrm{i}K')_{k+1,k} = f(v)(+\mathrm{i}) \exp(-\mathrm{i}\pi k\eta/K) \hat{S}_R(\mp, \beta)(v)_{k+1,k}. \tag{83}$$

Perform the similarity transformation

$$\bar{S}(\alpha, \beta)_{i,l} = A_{i,j} \hat{S}(\alpha, \beta)_{j,k} A^{-1}_{k,l}, \tag{84}$$

with

$$A_{k,l} = \delta_{k,l}(-\mathrm{i})^{k-1} \exp\left(\frac{\mathrm{i}\pi}{2K}(k-1)k\eta\right) a_1. \tag{85}$$

Then for $k < L$,

$$\bar{S}_R(\pm, \beta)(v + \mathrm{i}K')_{k,k+1} = f(v) \hat{S}_R(\mp, \beta)(v)_{k,k+1} \tag{86}$$

$$\bar{S}_R(\pm, \beta)(v + \mathrm{i}K')_{k+1,k} = f(v) \hat{S}_R(\mp, \beta)(v)_{k+1,k} \tag{87}$$

and for $k = L$,

$$\bar{S}_R(\pm, \beta)(v + \mathrm{i}K')_{L,1} = f(v) \exp\left[\frac{+\mathrm{i}\pi}{2}((2m_1 - 1)L + 2m_1)\right] \hat{S}_R(\mp, \beta)(v)_{k,k+1} \tag{88}$$

$$\bar{S}_R(\pm, \beta)(v + \mathrm{i}K')_{1,L} = f(v) \exp\left[\frac{-\mathrm{i}\pi}{2}((2m_1 - 1)L + 2m_1)\right] \hat{S}_R(\mp, \beta)(v)_{k+1,k}. \tag{89}$$

We find that if

$$\exp\left[\frac{-\mathrm{i}\pi}{2}((2m_1 - 1)L + 2m_1)\right] = 1, \tag{90}$$

$\hat{Q}_R$ satisfies the relation

$$\hat{Q}_R(v + \mathrm{i}K') = q^{-N/4} \exp\left(-\frac{\mathrm{i}\pi N v}{2K}\right) R \hat{Q}_R(v) \tag{91}$$

which is the same as (79) for $Q_{R,72}$. This happens only for

(I) even $m_1$ if $L = 4\times$ integer
(II) odd $m_1$ if $L = 2\times$ odd integer.

These are exactly those cases in which $\hat{Q}_R$ is singular. Like equation (79) equation (91) does not give the corresponding relation for the $Q$-matrix $\hat{Q}$.

In the following paragraph $\mathsf{Q}$ denotes either $Q_{72}$ or $\hat{Q}$.

We note that if the relation

$$\mathsf{Q}(v + \mathrm{i}\mathsf{K}') = q^{-N/4} \exp\left(-\frac{\mathrm{i}\pi N v}{2K}\right) R\mathsf{Q}(v) \tag{92}$$

were correct then it would follow that

$$q(v + \mathrm{i}\mathsf{K}')|q\rangle = q^{-N/4} \exp\left(-\frac{\mathrm{i}\pi N v}{2K}\right) q(v)R|q\rangle, \tag{93}$$

where $|q\rangle$ denotes an arbitrary eigenvector of $\mathsf{Q}(v)$ and $q(v)$ is its eigenvalue.

In other words: all eigenvectors of $\hat{Q}_(v)$ would be eigenvectors of $R$. It is however well known [5] that the eigenvectors of $Q_{72}(v)$ which are degenerate eigenvectors of the transfer matrix $T$ are generally not eigenvectors of $R$.

Equations (79) and (91) allow a coherent explanation of the fact that $Q_R$ is singular for one set of $\eta$ values and regular for another. Under the assumption that if $\mathsf{Q}$ exists there are eigenstates of $\mathsf{Q}$ which are not eigenstates of $R$, $Q_R$ cannot be regular if in case of $Q_{72}, m_1$ is even or in case of $\hat{Q}$ (90) is satisfied. This also explains naturally the observation that for fixed $L$ and sufficiently small $N$ $Q_R^{-1}$ exists always as then all states are singlets and (79) and (91) do not lead to contradictions when upgraded from $\mathsf{Q}_R$ to $\mathsf{Q}$.

Using the method used in this section it can easily be shown that always

$$\hat{Q}_R(v + 2\mathrm{i}K') = q^{-N} \exp(-\mathrm{i}N\pi v/K)\hat{Q}_R(v) \tag{94}$$

and consequently if $\hat{Q}_R^{-1}$ exists

$$\hat{Q}(v + 2\mathrm{i}K') = q^{-N} \exp(-\mathrm{i}N\pi v/K)\hat{Q}(v). \tag{95}$$

## 3. The properties of $\hat{Q}$ for $t = 0$

It follows from (95) that as shown for $Q_{72}$ in [5], $\hat{Q}(v)$ may be written as

$$\hat{Q}(v) = \hat{\mathcal{K}}(q; v_k) \exp(\mathrm{i}(n_B - \nu)\pi v/2K) \prod_{j=1}^{n_B} h\left(v - v_j^B\right)$$
$$\times \prod_{j=1}^{n_L} H(v - \mathrm{i}w_j)H(v - \mathrm{i}w_j - 2\eta) \cdots H(v - \mathrm{i}w_j - 2(L-1)\eta) \tag{96}$$

$$2n_B + Ln_L = N. \tag{97}$$

$n_B$ is the number of Bethe roots $v_k^B$ and $n_L$ is the number of exact $Q$-strings of length $L$. The sum rules (2.16)–(2.21) of [5] are also true for $\hat{Q}(v)$.

From (71) and

$$\hat{Q}_L(v)\hat{Q}_R(u) = \hat{Q}_L(u)\hat{Q}_R(v) \tag{98}$$

follows that

$$[S, \hat{Q}(v)] = 0. \tag{99}$$

Finally, we find numerically that the functional relation (3) which was originally conjectured in [5] is also satisfied for $\hat{Q}(v)$.

## 4. The matrix $\hat{Q}(v, t)$

We have shown that $\hat{Q}_R(v)$ satisfies the $T - \hat{Q}_R$ relation and found that it is not singular. But the proof of relation (31) failed for parameter $t \neq 0$. One finds numerically that (31) is in fact violated for systems large enough to allow degenerate eigenvalues of the transfer matrix. Therefore, the question arises whether $\hat{Q}_R(v, t)$ is useful at all. Surprisingly, we find numerically that for $\eta = 2mK/L$ and odd $L$ despite

$$\hat{Q}_L^{-1}(v_0, t)\hat{Q}_L(v, t) \neq \hat{Q}_R(v, t)\hat{Q}_R^{-1}(v_0, t) \tag{100}$$

both matrices

$$\hat{Q}^{(L)}(v, t) = \hat{Q}_L^{-1}(v_0, t)\hat{Q}_L(v, t) \qquad \text{and} \qquad \hat{Q}^{(R)}(v, t) = \hat{Q}_R(v, t)\hat{Q}_R^{-1}(v_0, t) \tag{101}$$

commute with the transfer matrix $T$. Furthermore, we find that in the cases studied $\hat{Q}^{(L)}(v, t)$ and $\hat{Q}^{(R)}(v, t)$ have the same eigenvalues. This means that there exists a matrix $A$ with $\hat{Q}^{(L)}(v, t) = A\hat{Q}^{(R)}(v, t)A^{-1}$ and consequently instead of (31)

$$\hat{Q}_L(v)A\hat{Q}_R(u) = \hat{Q}_L(u)A\hat{Q}_R(v) \tag{102}$$

should hold. A consequence of (100) is that

$$[\hat{Q}^{(R)}(v_1, t), \hat{Q}^{(R)}(v_2, t)] \neq 0 \tag{103}$$

as one needs (31) to prove that $Q$-matrices with different arguments commute (see 9.48.41 in [9]). We find that like $Q_{72}$ the matrix $\hat{Q}$ does not commute with $R$:

$$[R, \hat{Q}(v, t)] \neq 0, \tag{104}$$

but unlike $Q_{72}$ as a consequence of (100) does also not commute with $S$ for $t \neq 0$:

$$[S, \hat{Q}(v, t)] \neq 0. \tag{105}$$

This is possible because the degenerate subspaces of $T$ have elements with both eigenvalues $\nu' = 0, 1$ of $S$ if $\eta = 2m_1K/L$ and $L$ is odd.

These properties of $\hat{Q}^{(L)}(v, t)$ and $\hat{Q}^{(R)}(v, t)$ imply that they act as non-Abelian symmetry operators in all degenerate subspaces of the set of eigenvectors of $T$. We finally mention that whereas the six-vertex limit of $Q_{R,72}$ does not exist it exists for $\hat{Q}_R$. The limit of $\hat{Q}_R(v, t = \mathrm{i}K'/2)$ for elliptic nome $q \to 0$ is well defined. Using

$$\lim_{q\to 0} H(u \pm \mathrm{i}K'/2) = \exp(\mp\mathrm{i}(u - \pi/2)) \lim_{q\to 0} \Theta(u \pm \mathrm{i}K'/2) = 1, \tag{106}$$

one gets a regular limiting $\hat{Q}_R$-matrix. It has been checked numerically that the resulting $\hat{Q}$-matrix commutes with $T_{6v}$.

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## Appendix

The transfer matrix of the eight-vertex model is

$$T(v)|_{\mu,\nu} = \mathrm{Tr}\, W_8(\mu_1, \nu_1)W_8(\mu_2, \nu_2)\cdots W_8(\mu_N, \nu_N), \tag{A.1}$$

where in the conventions of (6.2) of [1]

$$\begin{aligned} W_8(1,1)|_{1,1} &= W_8(-1,-1)|_{-1,-1} = a = \rho\Theta(2\eta)\Theta(v-\eta)H(v+\eta) \\ W_8(-1,-1)|_{1,1} &= W_8(1,1)|_{-1,-1} = b = \rho\Theta(2\eta)H(v-\eta)\Theta(v+\eta) \\ W_8(-1,1)|_{1,-1} &= W_8(1,-1)|_{-1,1} = c = \rho H(2\eta)\Theta(v-\eta)\Theta(v+\eta) \\ W_8(1,-1)|_{1,-1} &= W_8(-1,1)|_{-1,1} = d = \rho H(2\eta)H(v-\eta)H(v+\eta). \end{aligned} \tag{A.2}$$

Relations used in the text. See e.g. [10]

$$\mathrm{sn}(u-v) = \frac{\mathrm{sn}(u)\,\mathrm{cn}(v)\,\mathrm{dn}(v) - \mathrm{sn}(v)\,\mathrm{cn}(u)\,\mathrm{dn}(u)}{1 - k^2\,\mathrm{sn}^2(u)\,\mathrm{sn}^2(v)} \tag{A.3}$$

$$H(v + \mathrm{i}K') = \mathrm{i}q^{-1/4}\exp\left(-\frac{\mathrm{i}\pi v}{2K}\right)\Theta(v) \tag{A.4}$$

$$\Theta(v + \mathrm{i}K') = \mathrm{i}q^{-1/4}\exp\left(-\frac{\mathrm{i}\pi v}{2K}\right)H(v). \tag{A.5}$$

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